Objective: Analyze and sketch the graph of a function without the use of any graphing devices or aids.

Concept 1: Find x and y-intercepts.

x-intercept - all points (x, 0) where the graph crosses the x-axis; known as zeros or roots of the function
y-intercept – all points (0, y) where the graph crosses the y-axis

To determine the intercepts:

1. **For x-intercepts: Let y = 0 and solve the equation for x;**
   utilizes algebraic manipulation; factoring; Zero-product property: If ab=0, then a=0 or b=0; Factor Thm and other techniques. In some cases it may prove almost impossible to determine and we may have to omit this calculation.

   Rational Root Thm: possible zeros of a function given by $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient (highest degree)

   Example: $6x^3 + 11x^2 - 3x - 2 = 0$
   possible values for p=2: $\pm 1,\ \pm 2$
   possible values for q=6: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$
   possible rational roots $\frac{p}{q}: \pm 1,\ \pm 2,\ \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}$

   The actual roots must fall in the domain of f.
   From this point may use either synthetic division or long division to find the actual roots of the desired polynomial.

   Integral Root Thm: If leading coefficient = 1, then any rational roots must be factors of the constant term.

   Example: $x^3 + 8x^2 + 16x + 5 = 0$
   $$\Downarrow \qquad\qquad \Downarrow$$
   $$\left.\begin{array}{c}\textit{leading}\\ \textit{coefficient}\end{array}\right\} = 1 \qquad \left.\begin{array}{c}\text{constant}\\ \text{term}\end{array}\right\} = 5 \quad \therefore \text{possible roots are } \pm 1,\ \pm 5$$

2. **For y-intercepts: Let x = 0 and solve the equation for y.**

Concept 2: Symmetry of the graph

To test for symmetry:

1. **y-axis symmetry – each point (x, y) on the graph also has the point (x,-y) on the graph; fold about the y-axis and match exactly.**
   Replacing x with a (-x) in the equation yields an equivalent equation
   Function is EVEN: f(-x) = f(x)

2. **x-axis symmetry – each point (x,y) on the graph also has the point (-x,y) on the graph; fold about the x-axis and match exactly.**
   Replacing y with a (-y) in the equation yields an equivalent equation

3. **origin symmetry – each point (x,y) on the graph also has the point (-x,-y) on the graph; graph to the right of the origin is rotated 180^0 it will fall exactly on the graph to the left of the origin.**
   Replacing BOTH x with a (-x) and y with a (-y) in the equation yields and equivalent equation.
   Function is ODD: f(-x) = - f(x)

Concept 3: Domain and Range

1. **<u>For Domain</u>: examine the function for restrictions such as division by zero or even roots which rule out negative numbers.**
2. **<u>For Range</u>: substitute possible x-values into original function to determine range ; fix x-y chart**

Concept 4: Continuity

Means no interruptions in the graph of the function; no holes, gaps, or jumps.
Continuity at a point: A function f is continuous at a point c if 3 conditions met:

1. **f(c) is defined – c must be in the domain and able to evaluate and get a real number solution**
2. $\lim_{x \to c} f(x)$ **- must show the left and right hand limits are equal**
3. $\lim_{x \to c} f(x) = f(c)$ **- must show or state these have SAME value**

Consideration is given to both removable and non-removable discontinuities.

Concept 5: Vertical, Horizontal & End Behavior, Slant Asymptotes

Look at the reduced function and apply following:

V.A. – at values that make reduced function's denominator = 0
H.A. – degree N = degree D or degree N < degree D
S.A. – degree N = degree D + 1
Where N $\to$ numerator
D $\to$ denominator

Vertical asymptotes occur with rational functions (has both a numerator and denominator) and exist at values that make the denominator zero.
To determine vertical asymptotes:

1. Check to see if function can be reduced; cancelled factors are holes
2. Set the denominator equal to zero.
3. Factor the denominator or use other techniques to solve for x.
4. These x-values represent the location of the vertical asymptotes.
5. Express vertical asymptotes as equations. VA: x = (some number)

Horizontal asymptote – occurs if lim f(x) approaches b as x approaches $\pm\infty$

- occurs when degree of denominator = degree of numerator OR degree of denominator > degree of numerator
- to find evaluate the limit at ∞ (as x approaches $\pm\infty$)
- may also use long division
- use technique of dividing both numerator and denominator by the highest power of x in the denominator
- keep in mind $\frac{1}{x} \to 0$ as $x \to \infty$
- expressed as an equation of the form y = b
- graph may cross a horizontal asymptote

To determine horizontal asymptotes and end behavior of a function:

1. Check degree of numerator and denominator first
2. Find both: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$
3. Express these values in equation form. HA: y = (some number)

To determine Slant asymptotes:

1. Check the degree of numerator and denominator first; degree N must equal degree D + 1
2. Use long division to find
3. Express quotient found in long division w/o remainder in equation form. SA: y = (some algebraic expression)

To determine End Behavior of the function:

1. Find both: $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

Concept 6: Differentiability

A function is differentiable at x if its derivative exists at x and the function is differentiable on an <u>open</u> interval (a, b) if it is differentiable at <u>every point in the interval</u>.

Three key points to remember:

1. **The derivative of a function is a function itself.**
2. **The derivative generates a slope formula which enables you to determine the slope at any point on the curve.**
   Of particular interest will be points that yield a slope of zero, one type of critical values.
3. **a) If a function is differentiable at x = c, then it is continuous at x=c; that is, differentiability implies continuity.**
   Places where vertical asymptotes, sharp corners, peaks, or oscillate are areas where the function is NOT differentiable.
   b) If a function is NOT continuous at c, then it is NOT differentiable at c. Keep in mind the 3 conditions for continuity.

Concept 7: Increasing and Decreasing Function

As the values of x move to the right a function is said to be increasing if the graph moves upward; constant if it follows a horizontal path; and decreasing if its graph moves downward.

Formal Definition:

1. **Increasing: A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$**

2. **Decreasing: A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$**

3. **Constant: A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) = f(x_2)$**

Remember that the derivative is a slope formula; $f'(x) = m$ (slope of tangent line). From our knowledge of slope we know a positive slope indicates the line is increasing, negative slope indicates the line is decreasing, and zero slope indicates a horizontal line.

Formal Definition: (First Derivative Test)
Let f be a function that is continuous on the closed interval [a, b] and differentiable on the open interval (a, b)

1. **If f'(x) > 0 for all x in (a, b), the f is increasing on [a, b]**
2. **If f'(x) < 0 for all x in (a, b), the f is decreasing on [a, b]**
3. **If f'(x) = 0 for all x in (a, b), the f is constant on [a, b]**

Critical numbers – values of x that make the function undefined; make the first derivative = 0 or infinite (DNE); or inflection points- where the second derivative changes sign

<u>Critical point THM</u>: Let f be defined on an interval I containing the point c. If f(c) is an extreme value, then c must be a critical point; that is either c is:

1. **an end point of I;**
2. **a stationary point of f; point where f'(c) = 0**
   Graphically: the graph of f levels off – tangent line will be horizontal at this point
3. **a singular point of f; point where f'(c) does not exist**
   Graphically: point where the graph of f has a sharp corner, vertical tangent line, or perhaps takes a jump, or near where the graph wiggles very badly. Extreme values can occur at singular point, thought in practical problems this is quite rare

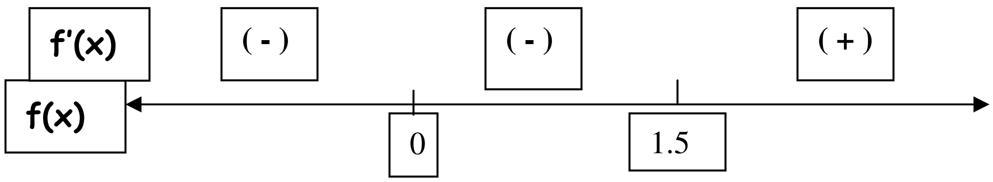
Guidelines for finding intervals where function is increasing/decreasing:

1. **Take the first derivative.**
2. **Set first derivative =0 and solve for x.**
3. **State all critical values: where x makes f'(x) = 0 AND where f(x) or f'(x) are undefined**
4. **Draw a straight line representing the x-values of f(x), label f and mark with all critical values in numerical order; make notes of undefined values. This creates intervals over which we evaluate f'(x).**
5. **Above the line label f'(x). Pick a test value within each of these intervals.**
6. **Substitute the test value into the first derivative and solve.**
7. **Determine the sign of the first derivative & mark results at f'(x)**
   (+) positive → function increasing
   (-) negative → function decreasing
8. **Interpret results in terms of the function.**

Example1: $f(x) = x^4 - 2x^3$ **Domain:** $(-\infty, \infty)$

$$f'(x) = 4x^3 - 6x^2$$
$$0 = 4x^3 - 6x^2$$
$$0 = 2x^2(2x - 3) \rightarrow 2x^2 = 0 \text{ or } 2x - 3 = 0$$
$$x = 0 \text{ or } x = 1.5$$

Critical values: 0, 1.5



Test values: x=-1 , x=1, x=2 NOTE: easier to used factored form of f'

$f'(-1) = 2(-1)^2 [2(-1)-3] = 2(-5) = (-) \rightarrow$ **decreasing**

$f'(1) = 2(1)^2 [2(1)-3] = 2(-1) = (-) \rightarrow$ **decreasing**

$f'(2) = 2(2)^2 [2(2)-3] = 8(1) = (+) \rightarrow$ **increasing**

RESULTS: the function is decreasing on $(-\infty, 0)$ and $(0, 1.5)$
the function is increasing on $(1.5, \infty)$

Concept 8: Critical values – maximum/minimum points

When the intervals on which a function is increasing or decreasing have been determined it is a relatively easy task to locate the relative extrema – maximum and minimum points - of the function.

A relative maximum exists when the interval to the left of the considered point is increasing and to the right of that point is decreasing.

Similarly, a relative minimum exists when the interval to the left of the considered point is decreasing and the interval to the right of that point is increasing.

Formal Definition: First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval containing c. If f is differentiable on the interval, except possibly at c, then f(c) can be classified as follows:

1. **f(c) is a relative minimum of f if f'(x) changes from negative to positive at c.**
2. **f(c) is a relative maximum of f if f'(x) changes from positive to negative at c.**
3. **If f'(x) does not change sign at c, then f(c) is NEITHER a relative maximum or a relative minimum.**

Looking at the previous example1: $f(x) = x^4 - 2x^3$ we see there is a first derivative sign change at 1.5. Using the First Derivative Test f' is negative to the left and positive to the right of 1.5, therefore 1.5 is a maximum.
The coordinates of the maximum are found by substituting 1.5 into f(x):

$$f(\frac{3}{2}) = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3$$
$$= \frac{81}{16} - 2\left(\frac{27}{8}\right)$$
$$= -\frac{27}{16}$$

Therefore the maximum occurs at $\left(\frac{3}{2}, \frac{-27}{16}\right)$

Concept 9: Concavity

Concave up – portion of the graph over an interval that opens up (smiley face)
Concave down – portion of the graph over an interval that opens down (frown face)
These concavities appear in the intervals where the function is increasing or decreasing.

Formal definition of Concavity

Let f be differentiable on an open interval. The graph of f is concave up on the interval if f'(x) is increasing on the interval and concave down if f'(x) is decreasing on the interval. Note we are looking at how the first derivative is acting.

Second Derivative Test for Concavity

Let f be a function whose second derivative, f''(x), exists on an open interval.

1. **If $f''(x) > 0$ for all x in the open interval, then the graph of f is concave up on that interval.**
2. **If $f''(x) < 0$ for all x in the open interval, then the graph of f is concave down on that interval.**
3. **If $f''(x) = 0$ for all of the values of x on the interval, then f is linear; concavity is not defined for a line. A line is neither concave up nor concave down.**

Remember: a point of inflection exists when a graph moves from concave up to concave down or vice versa; not just because the 2nd derivative =0.

Guidelines for finding the intervals on which a function is concave up/down:

1. **State the domain of f**
2. **Take the first derivative**
3. **Take the second derivative and state its domain**
4. **Determine when f"(x) = 0 or DNE; these values are possible inflection points and provide test intervals**
5. **Draw line and label as f" and mark inflection points**
6. **Choose a test number within each interval**
7. **Substitute the test number into the second derivative, f" to determine whether result is (+) or (-); mark findings on line**
8. **State conclusions with regard to concavity using Second Derivative Test**
9. **For x-values that are true inflection points, substitute the x-value into the original function to determine corresponding y-value so as to express at point (x,y)**

<u>Example2</u> applying Concepts 1-9: (order of use depends on function)

Function: $f(x)=\dfrac{x^4+1}{x^2}$

1. **Being a rational function, we see $x^2 \neq 0$; therefore x = 0 is not in domain. For the numerator not restrictions exist. Domain: $(-\infty, 0) \cup (0, \infty)$ We will come back to the Range as future results will give more insight.**

2. **Asymptotes: Since x = 0 makes the function undefined; VA: x=0**
   Examining the function using the following test:
   H.A. – degree N = degree D or degree N < degree D
   S.A. – degree N = degree D + 1
   we find neither applies; therefore no HA or SA exists.

3. **End Behavior: use limits**

$$\lim_{x\to\infty}\frac{x^4+1}{x^2}=\frac{\infty+1}{\infty}=\frac{+\infty}{+\infty}\Rightarrow+\infty$$

$$\lim_{x\to-\infty}\frac{x^4+1}{x^2}=\frac{\infty+1}{\infty}=\frac{+\infty}{+\infty}\Rightarrow+\infty$$

We see the function approaches $+\infty$ as x approaches both $-\infty$ and $+\infty$

Example2 con't

4. Intercepts:

Let $x = 0$. $f(0) = \dfrac{0+1}{0} \Rightarrow$ undefined $\Rightarrow$ no x-intercepts

Let $y = 0$ and solve for x. $0 = \dfrac{x^4+1}{x^2}$ This quantity will only be =0 when numerator = 0

$$0 = x^4 + 1$$

$$-1 = x^4$$

$$\sqrt[4]{-1} = x \Rightarrow \text{imaginary roots} \quad \therefore \text{no y-intercepts}$$

5. Find Critical numbers

$$f(x) = \frac{x^4+1}{x^2} = (x^4+1)x^{-2}$$

$$f(x) = x^2 + x^{-2}$$

$$f'(x) = 2x + (-2)x^{-3}$$

$$= 2x - \frac{2}{x^3}$$

$$= \frac{2x^4-2}{x^3}$$

$$f'(x) = \frac{2(x^4-1)}{x^3} \Rightarrow \text{undefined when } x^3 = 0 \text{ or } x = 0$$

$$0 = 2(x^4-1)$$

$$1 = x^4$$

$$\sqrt[4]{1} = x$$

$$x = \pm 1$$

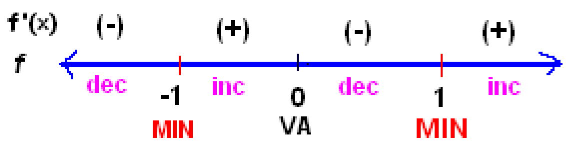
Therefore critical numbers are: x =-1, 0, 1

NOTE: Remember to include any values for which the original function is undefined.

Example2 con't

6. Increasing/decreasing and min/max values

Now that we have the critical values draw and label a line representing f and label f'(x) above. Use test values to see what is happening.



Work for above results:

Test values: x=-2; x=-1/2; x=1/2; x=2

$$f'(x)=\frac{2(x^4-1)}{x^3}$$

$$f'(-2)=\frac{2[(-2)^4-1]}{(-2)^3}=\frac{2(15)}{-8}=(-)$$

$$f'\left(\frac{-1}{2}\right)=\frac{2\left[\left(-\frac{1}{2}\right)^4-1\right]}{\left(-\frac{1}{2}\right)^3}=\frac{2\left(\frac{1}{16}-1\right)}{-\frac{1}{8}}=\frac{(-)}{(-)}=(+)$$

$$f'\left(\frac{1}{2}\right)=\frac{2\left[\left(\frac{1}{2}\right)^4-1\right]}{\left(\frac{1}{2}\right)^3}=\frac{2\left(\frac{1}{16}-1\right)}{\frac{1}{8}}=\frac{(-)}{(+)}=(-)$$

$$f'(2)=\frac{2[(2)^4-1]}{(2)^3}=\frac{2(15)}{8}=(+)$$

Results: f(x) is increasing on (-1, 0) and (1, ∞)
f(x) is decreasing on (- ∞, -1) and (0, 1)

Example2 con't

Applying the First Derivative Test to above results shows that x = -1 & x = 1 are minimums. Now we have to find their corresponding y-values.

$$f(x) = \frac{x^4+1}{x^2}$$

$$f(-1) = \frac{(-1)^4+1}{(-1)^2} = \frac{2}{1} = 2$$

$$f(1) = \frac{(1)^4+1}{(1)^2} = \frac{2}{1} = 2$$

Therefore minimums occur at (-1, 2) and (1, 2)

7. **End behavior and asymptotic behavior**
   We need to see how the function is behaving as x approaches ±∞ and x approaches the VA on each side. For this we go back to the original function.

$$\lim_{x\to\infty} \frac{x^4+1}{x^2} = \frac{\infty+1}{\infty} \to \infty$$

$$\lim_{x\to-\infty} \frac{x^4+1}{x^2} = \frac{(-\infty)^4+1}{(-\infty)^2} = \frac{\infty+1}{\infty} \to \infty$$

$$\lim_{x\to0^+} \frac{x^4+1}{x^2} \Rightarrow \frac{\left(\frac{1}{10}\right)^4+1}{\left(\frac{1}{10}\right)^2} = \frac{+}{+} = (+) \Rightarrow \infty$$

$$\lim_{x\to0^-} \frac{x^4+1}{x^2} \Rightarrow \frac{\left(-\frac{1}{10}\right)^4+1}{\left(-\frac{1}{10}\right)^2} = \frac{+}{+} = (+) \Rightarrow \infty$$

We see that at the left end of the graph y grows w/o bound; at the right end of the graph y grows w/o bound; and as x approaches the VA from the right y grows w/o bound; as x approaches the VA from the left y grows w/o bound.

8. Range

Using the above finding from steps 6 & 7 gives insight into the range. From step 6 we find the minimum value y can attain is 2; from step 7 we see there is no maximum value. Thus the Range: $(2, \infty)$.

9. Inflection points & concavity

For this we need the second derivative of the function. Return to our previous work of finding the first derivative:

$$f(x) = \frac{x^4+1}{x^2} = (x^4+1)x^{-2}$$

$$f(x) = x^2 + x^{-2}$$

$$f'(x) = 2x + (-2)x^{-3} \quad \Leftarrow \text{use this form to take 2nd derivative}$$

$$f'(x) = \frac{2(x^4-1)}{x^3}$$

For the second derivative:

$$f'(x) = 2x + (-2)x^{-3}$$

$$f''(x) = 2 + 6x^{-4}$$

$$= 2 + \frac{6}{x^4}$$

$$f''(x) = \frac{2(x^4+3)}{x^4} \qquad \text{set = 0 and solve to find inflection pts}$$

$$\frac{2(x^4+3)}{x^4} = 0 \Rightarrow x \neq 0 \quad \textit{and} \text{ is 0 whenever numerator =0}$$

$$2(x^4+3) = 0$$

$$x^4 + 3 = 0$$

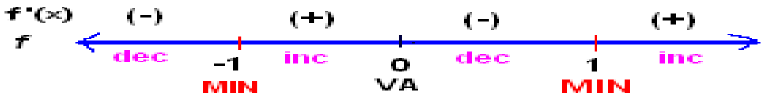
$$x^4 = -3$$

$$x = \sqrt[4]{-3} \Rightarrow \textit{imaginary roots}$$

Possible inflection pts: x=0

The function has a VA at x=0 so 0 is not in domain of f; $\therefore$ no points of inflection

Therefore we go back to looking at sign changes found for increasing/decreasing intervals



Looking at these results we see that the first derivative changes signs at -1 and 1; in each case the sign is from (-) to (+) indicating concave up for both. **<u>Be careful to omit 0 here as it is not in the domain of the function.</u>**

All that is left for Concepts 1-9 is to summarize our findings in an organized format and sketch the function.

Example2: Graph Analysis using Concepts 1-8

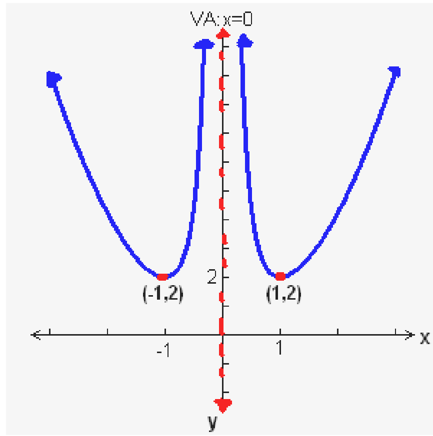
Function: $f(x)=\dfrac{x^4+1}{x^2}$

Domain: $(-\infty\ ,\ 0) \cup (0,\ \infty)$

Range: $(2,\infty)$
VA: x = 0
HA: none
SA: none
x-intercept: none
y-intercept: none
CN: x =-1, 0, 1
Inflection points: none
minimum: (-1, 2) and (1, 2)
maximum: none
increasing on: (-1, 0) and $(1,\ \infty)$
decreasing on: $(-\infty,\ -1)$ and (0, 1)
concave up on $(-\infty,\ 0)$ and $(0,\ \infty)$

Now we must use this information to draw and label as accurate SKETCH of the graph as possible.

Draw the x-y axis and label.
Add any VA or HA asymptotes using dashed lines and label.
Plot all points for intercepts, max/min, and inflection points; label each.
Starting at the left of the graph and using increasing/decreasing and concavity information, sketch the graph.



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