

C-1.2 Finding Limits Graphically & Numerically

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Objectives: Estimate a limit using a numerical or graphical approach.
Learn different ways a limit can fail to exist.
Introduce formal definition of limit and proof notation/concepts

Definition: Intuitive Meaning of Limit & notation

To say that $\lim_{x \rightarrow c} f(x) = L$ means that when x is near but different from c then $f(x)$ is near L .

$\lim_{x \rightarrow c} f(x) = L$ also read as "the limit of $f(x)$ as x tends to c is L "

or "as x approaches c , $f(x)$ approaches L "

or "when x is close to c but different from c ,
 $f(x)$ is close to L "

Notice we do not require anything at c . The notion of a limit is associated with the behavior of a function **near c , not at c** ; in the neighborhood of c . The existence or nonexistence of $f(x)$ at c has no bearing on the existence of the limit of $f(x)$ as x approaches c .

Two natural categories:

1. **Right-handed limit:** means that when x is near but on the right of c then $f(x)$ is near L

Denoted: $\lim_{x \rightarrow c^+} f(x) = L$

2. **Left-hand limit:** means that when x is near but on the left of c then $f(x)$ is near L

Denoted: $\lim_{x \rightarrow c^-} f(x) = L$

These are referred to as one-sided limits.

THM: The relationship between the limit of f as x approaches c and the two one-sided limits of f as x approaches c is

$$\lim_{x \rightarrow c} f(x) = L \quad \text{where } L \text{ must be a real number}$$

if and only if

$$\lim_{x \rightarrow c^+} f(x) = L \quad \text{AND} \quad \lim_{x \rightarrow c^-} f(x) = L$$

This implies the limit as $x \rightarrow c$ x is **UNIQUE**; cannot have 2 different limits as $x \rightarrow c$.

Class video: Tutorials for the Calculus Phobe: www.calculus-help.com/tutorials

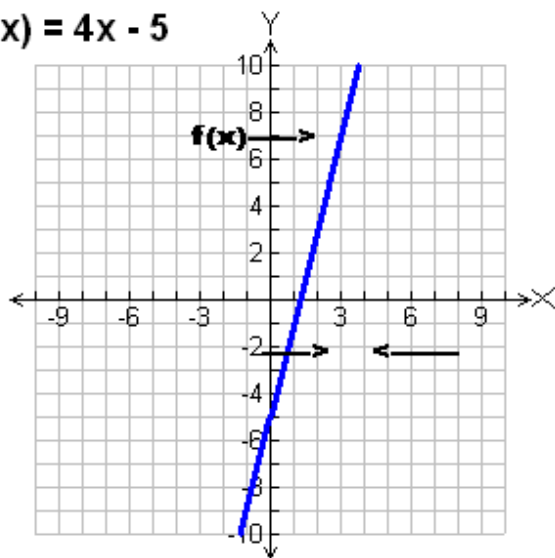
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Class examples:

1. Find $\lim_{x \rightarrow 3} (4x - 5)$

$f(x) = 4x - 5$

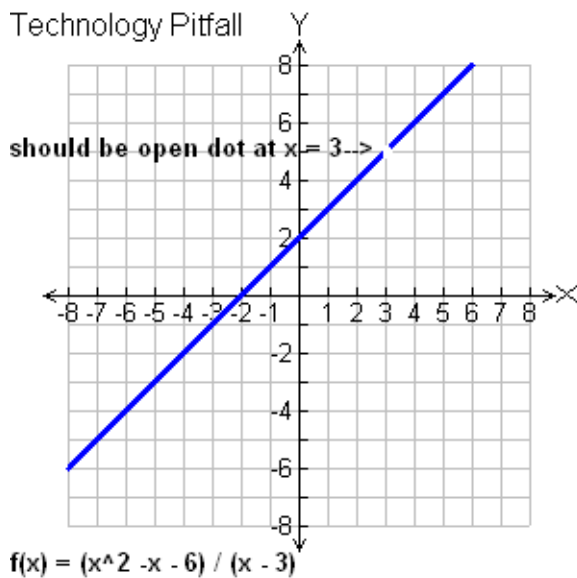


x	y1(x) 4x-5
2	3
2.1	3.4
2.2	3.8
2.3	4.2
2.4	4.6
2.5	5
2.6	5.4
2.7	5.8
2.8	6.2
2.9	6.6
3	7
3.1	7.4
3.2	7.8
3.3	8.2

$\lim_{x \rightarrow 3} (4x - 5) = 7$

2. Find $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$ notice here $x \neq 3$

Technology Pitfall

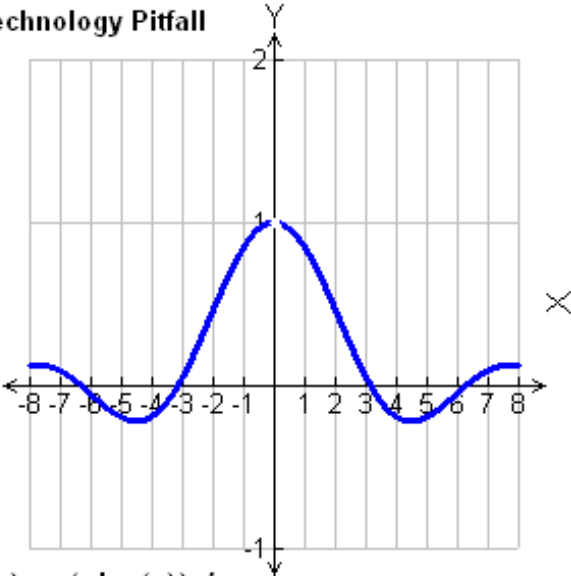


x	y1(x) (x ² -x-6) x-3
2.91	4.91
2.92	4.92
2.93	4.93
2.94	4.94
2.95	4.95
2.96	4.96
2.97	4.97
2.98	4.98
2.99	4.99
3	undef
3.01	5.01
3.02	5.02
3.03	5.03
3.04	5.04

$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} = 5$

3. Find $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Technology Pitfall



$$f(x) = (\sin(x)) / x$$

x	y1(x) (sin)
-0.8	0.8967
-0.7	0.92031
-0.6	0.94107
-0.5	0.95885
-0.4	0.97355
-0.3	0.98507
-0.2	0.99335
-0.1	0.99833
0	undef
0.1	0.99833
0.2	0.99335
0.3	0.98507
0.4	0.97355
0.5	0.95885

NOTE: not all limits exist! When this happens must write out "does not exist" as your answer; later on use DNE.

When Limits fail to exist:

1. When the one-sided limits are different

$$\lim_{x \rightarrow c^+} f(x) \neq \lim_{x \rightarrow c^-} f(x)$$

2. $f(x)$ increases or decreases without bound ($-\infty$ or ∞) as x approaches c ; $f(x)$ approaches a vertical asymptote; does not approach a real number
3. $f(x)$ oscillates (bounces back & forth) between two fixed values as x approaches c (too many wiggles)
4. No limit at a jump ---such as the step function

Must be careful and aware of the pitfalls of calculators!!

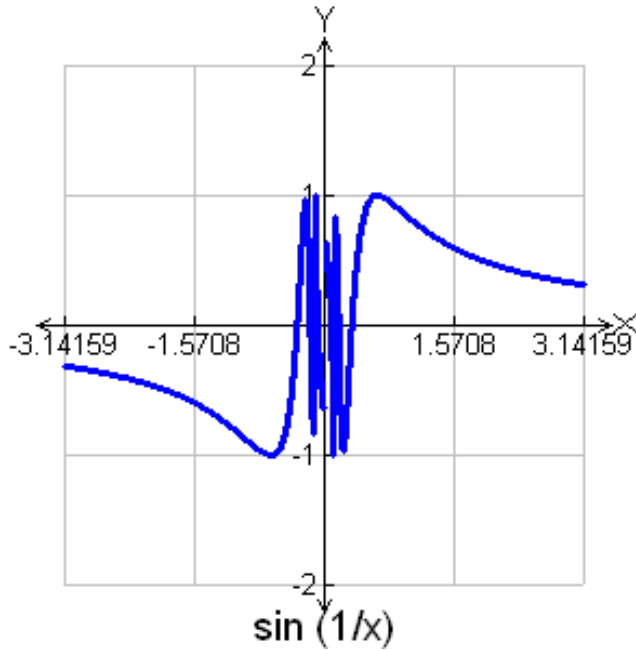
Compare calculator graph to ACTUAL graph and table of values.

Here both the graphical and numerical approach are important along with mathematical knowledge.

Class examples:

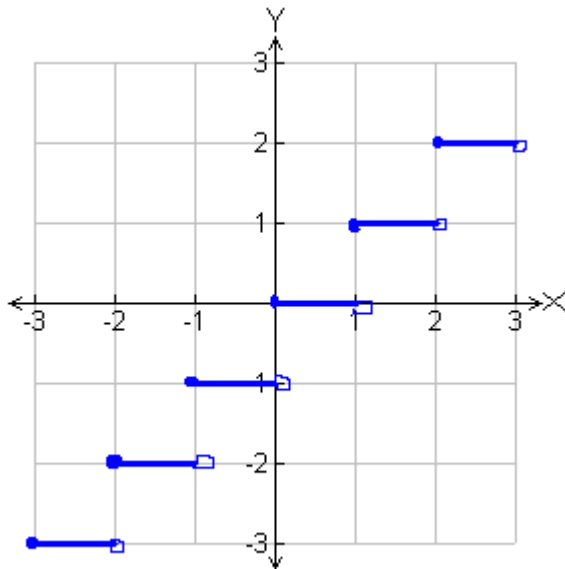
1. Too many wiggles!

Find $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$



x	y1(x) sin(1/
-1	-0.8415
-0.9	-0.8962
-0.8	-0.949
-0.7	-0.9899
-0.6	-0.9954
-0.5	-0.9093
-0.4	-0.5985
-0.3	0.19057
-0.2	0.95892
-0.1	0.54402
0	undef
0.1	-0.544
0.2	-0.9589
0.3	-0.1906

2. No limit at a jump! Find $\lim_{x \rightarrow 2} [x]$



x	y1(x) int(x)	x	y1(x) int(x)
-3	-3	1.3	1
-2	-2	1.4	1
-1	-1	1.5	1
0	0	1.6	1
1	1	1.7	1
2	2	1.8	1
3	3	1.9	1
4	4	2	2
5	5	2.1	2
6	6	2.2	2
7	7	2.3	2
8	8	2.4	2
9	9	2.5	2
10	10	2.6	2

Be sure to set table values to see the neighborhood of value approaching.

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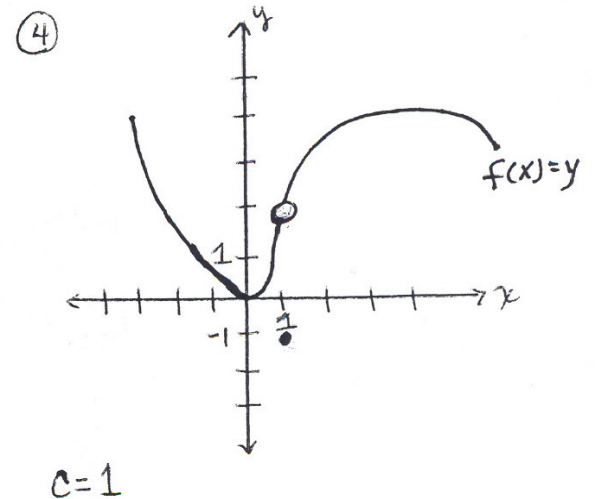
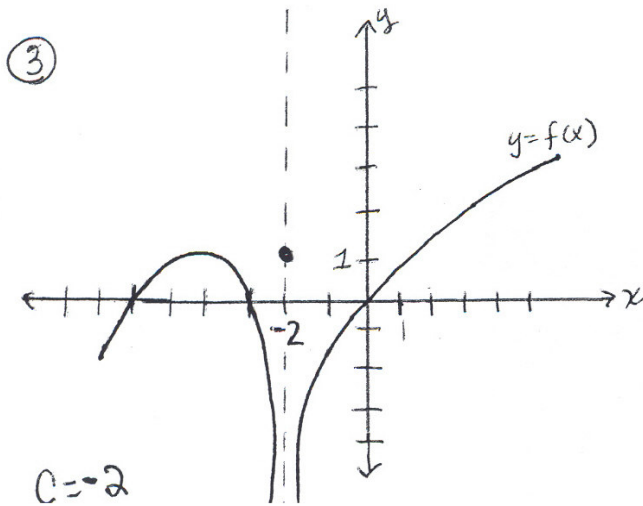
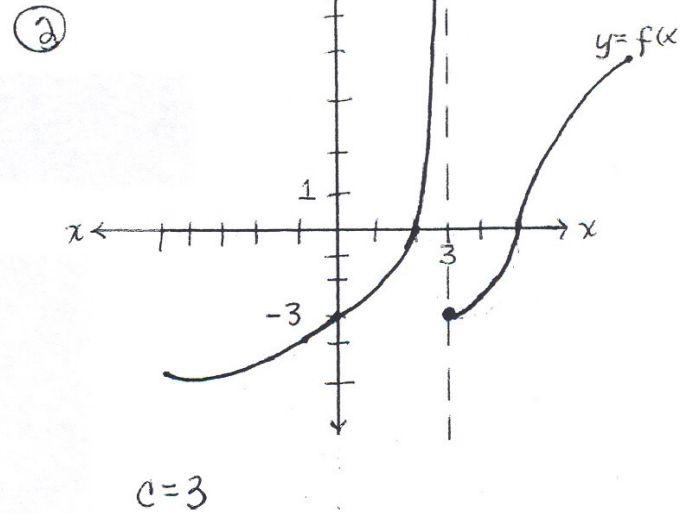
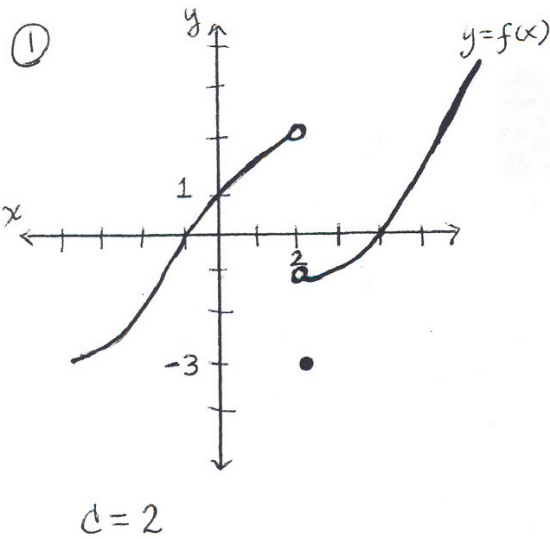
Graphical approach

Use the graph of f to find
 d) $f(c)$

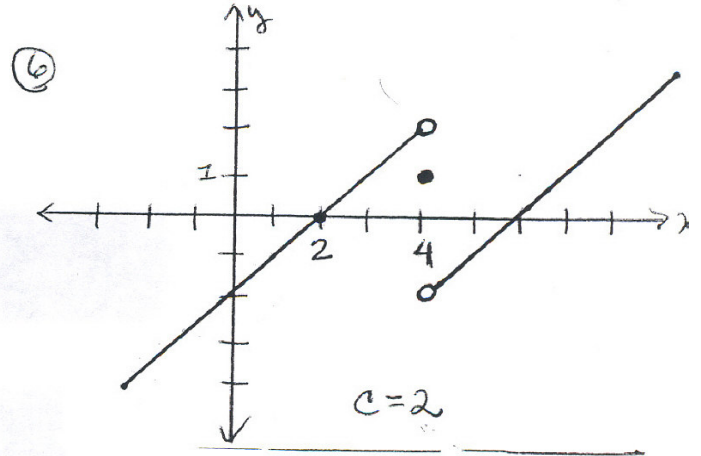
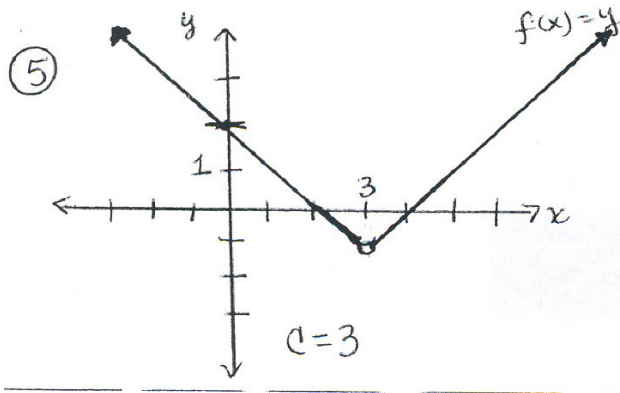
a) $\lim_{x \rightarrow c^-} f(x)$

b) $\lim_{x \rightarrow c^+} f(x)$

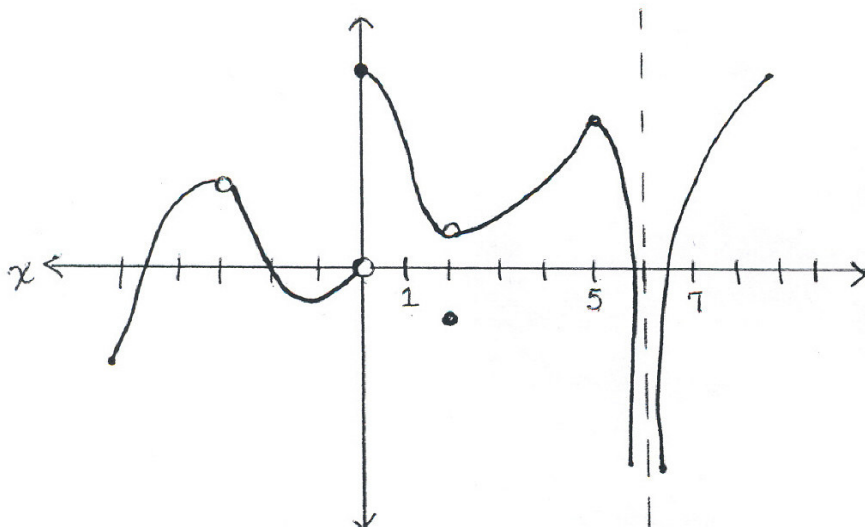
c) $\lim_{x \rightarrow c} f(x)$



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State the values of c for which $\lim_{x \rightarrow c} f(x)$ does NOT exist



Definition: Formal or precise def. of Limit

Let f be a function defined on an open interval containing c (except possibly at c) AND let L be a real number, then the

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

OR

This is to say that $\lim_{x \rightarrow c} f(x) = L$ means that for each given $\varepsilon > 0$

(no matter how small) there is a corresponding $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \text{ provided } 0 < |x - c| < \delta; \text{ that is}$$

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Specific examples covered in separate class.

It is very important that you know this definition as we will be doing some formal proofs of limits using it.

Mathematical Proof symbols:

Symbol	Meaning
\Rightarrow	implies
\Leftrightarrow	if and only if
\exists	there is (there are, there exists)
\forall	for all (for each, for any, for every)
\ni	such that; for which
\wedge	and
\vee	or
\therefore	therefore
\square	Q.E.D. (which was to be demonstrated) Latin: Quod erat demonstrandum (which was to be proved)

Symbol	Meaning
\in	is an element (member) of
\notin	is not an element of
\subseteq	is a subset of
\emptyset	Empty or null set
\sim	not
\neg	not

In mathematics, a statement is a sentence that is either true or false. A proof is a convincing argument expressed in the language of mathematics. It should contain enough mathematical details to be convincing to the person(s) to whom it is addressed. Different types exist.

Mathematical Induction - a technique for proving that for every integer $n \geq$ some initial integer n_0 , some statement $P(n)$ is true.

To do so, one must show that $P(n_0)$ is true. Then assume that $P(n)$ is true and show that $P(n+1)$ is true by relating $P(n+1)$ to $P(n)$.

References

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